



# PHD COURSE IN MATHEMATICAL MODELS AND METHODS IN ENGINEERING

Chair:  
Prof. Michele Correggi

Mathematics is everywhere, represented by equations. Between the atmosphere and the wing of a spaceship, in the blood flowing in an artery, on the demarcation line between ice and water at the poles, in the motion of the tides, in the charge density of a semiconductor, in the compression algorithms of a signal sending images from outer space. The equations represent real problems. The Mathematical Engineer can see and understand the nature of these equations, and can develop models in order to understand their relevant qualities and solve real problems. This PhD program aims at training young researchers by providing them with a strong mathematical background and with ability to apply their knowledge to the solution of real-world problems that arise in various areas of science, technology, industry, finance, management, whenever advanced methods are required in analysis, design, planning, decision and control activities. PhD students carry out their research both in the development of new mathematical methods and in the implementation and improvement of advanced techniques in connection with specific contexts and applications.

The Faculty of the PhD program is responsible for the organization of the training and research activities of the PhD students. Decisions of the Faculty comply with the requirements and standards of the Doctoral School of the Politecnico di Milano. A Chairman is elected within the Faculty, for representative and coordination activities. Admission of students to the PhD program is decided after examination of the candidates. Students applying to our program must provide their CV, along with reference and motivation letters. After admission, each student is assigned a tutor. The tutor is a member of the Faculty who assists the student in the early stages of his career, especially in the choice of the courses and in identifying a thesis advisor.

The PhD program has a duration of three years. Activities include: soft skills courses; specialized courses; research training, including seminars, tutoring activity, participation to workshops/conferences, and scientific publications; development of a doctoral thesis.

At the end of each academic year, the PhD students report to the Faculty about their activity. The students report about attendance of courses and exams (and the corresponding grades), participation in various scientific

activities (seminars, conferences, summer schools etc.), planning and intermediate results on their research project and preparation of the PhD thesis, and any other relevant activity. At the annual meeting the students also receive a grade by the Faculty. A negative grade may entail repetition of the current year of doctoral study (with suspension of the grant, if any) or exclusion from the PhD program, depending on the Faculty's decision. Mobility of PhD students to other institutions is strongly encouraged and financial support is provided to this purpose.

Among others, let us mention some typical types of professional skills and possible occupations of the graduated Doctors: analytic and numerical treatment of differential models for physical and industrial problems, quantitative methods in finance and risk management, operations research and optimisation, statistical modelling and data analysis.

Placement of graduated Doctors is expected in the following positions: research and development divisions of businesses, businesses involved in innovative design activities, financial institutions such as banks or insurance companies, public or private research centres, public and/or governmental agencies for social, economical, scientific study, planning or evaluation, Universities.

Since the PhD program in Mathematical Models and Methods in Engineering (formerly, Mathematical Engineering) has been active since the year 2001, we expect that a larger number of institutions and businesses will soon become more and more aware of the professional skills and expertise of graduated doctors.

**Grants funded by external partners:** IIT, Leonardo Spa, Pirelli Tyre Spa, ENI Spa, RSE, Intellico

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## A HIGH ORDER ACOUSTIC SOLVER FOR AEROACOUSTIC PROBLEMS

Alberto Artoni – Supervisors: Nicola Parolini, Daniele Rocchi

Aeroacoustics is the branch of acoustics that studies the noise induced by flows. Nowadays regulation aiming to reduce noise pollution and increase the liveability of our communities, require proper design tools to best capture the aeroacoustic phenomenon, critical in air and ground transports. We introduce a high-fidelity and high dimensionality high order Discontinuous Galerkin Spectral Element Method (DGSEM) to solve the segregated aeroacoustic problem. The segregated strategy first solves the flow problem based on employing Finite Volume (FV) methods with the library OpenFOAM, an open-source library specialized in CFD applications. Then, the flow solution is employed to compute the sound source for the acoustic problem via a suitable post processing approach. This requires suitable projection operations between the non-nested three-dimensional meshes that preserve the overall accuracy of the segregated approach. Finally, we solve an inhomogeneous wave equation to propagate the noise generated by the flow based on employing an in-house software, AeroSPEED based on a high order Discontinuous

Galerkin Spectral Element method. The Spectral Element method provides low the dissipation and dispersion errors, and is enhanced with the Discontinuous Galerkin method, allowing non conformal spaces easing the mesh generation. The segregated problem is coupled through a novel intergrid projection method that computes explicitly the intersection between the two (non-nested) computational grids of the flow and acoustic problems. We prove and demonstrate through extensive testing the accuracy of the intersection algorithm, and we verify the scalability of our implementation. The fully discrete acoustic problem has been analysed by considering the presence of the projected source term. We first analyse the projection error proving and verifying a priori estimates in the case of piecewise constant flow solution and we generalise it by considering high order reconstruction of the finite volume flow solution. Then, an a priori error analysis is carried out and verified for the fully discrete acoustic problem where the source term is a given projected piecewise constant or linearly reconstructed function. The analysis further highlights how

is crucial to map properly the sound source term from the flow grid onto the acoustic grid. We apply our computational strategy to a wide number of benchmarks. First, we assess the capabilities of our acoustic solver that leads to remarkable performance with respect to commercial software in terms of speed and accuracy. We also simulate a real test case that generates a monopole sound source. The numerical solution has been compared with the experimental data. Finally, the proposed computational framework is applied to aeroacoustics benchmarks. We considered flow problems where the solution was given, computed from a laminar problem or computed from turbulent problems. We conclude by showing the

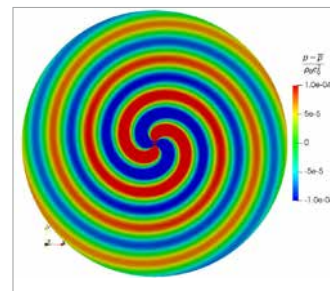


Fig. 1 - The Corotating Vortex Pair is an analytical aeroacoustic case where the sound source is given by an inviscid flow solution, while the acoustic solution is a rotating quadrupole.

three-dimensional capabilities of the solver solving the aeroacoustic noise around a side view mirror of a vehicle. Our results have been validated with the benchmark results available in the literature.

# POLYTOPAL DISCONTINUOUS GALERKIN METHODS FOR WAVE PROPAGATION PHENOMENA IN THERMO-POROELASTIC MEDIA

**Stefano Bonetti** - Supervisor: Paola F. Antonietti

Co-Supervisor: Michele Botti

The study of wave propagation phenomena in heterogeneous media has various fields of applications, from structural mechanics to biology. This thesis inspects the wave propagation problem in the context of geophysics and seismology. Among the most important applications that would exploit the numerical modeling of this phenomenon, it is worth mentioning geothermal energy production and greenhouse gas sequestration – which are crucial for energy sustainability – or the study of thermo-elastic seismic energy release as a source mechanism for volcanic earthquakes. Induced earthquakes by human soil exploitation activities represent another important application.

To study this problem, it is of particular interest to investigate the thermo-hydro-mechanical (THM) coupling, which refers to the coupled interactions between temperature, fluid flow, and mechanical deformations. It is common in the literature to study the subsurface by looking at the coupling between the fluid problem and the mechanics. However, the additional coupling with thermal effects in this case is of crucial importance, as in the

applications of interest strong temperature gradients are present. For example, geothermal power generation processes are characterized by the extraction of fluid from the subsurface at high temperatures and the injection of fluid into the subsurface at temperatures significantly below the temperature of the subsurface itself. Consequently, the hydro-mechanical response depends not only on the injection of fluid into the subsurface but also on the thermal-type phenomena that are triggered by the strong temperature gradients. As the THM coupling finds application in different contexts, in the thesis two different regimes for the model are considered.

In the first part of the thesis, the quasi-static THM model problem is introduced. It consists of three equations: the energy conservation equation, which is solved for the temperature, the mass conservation equation for the pressure, and the momentum conservation equation for the solid displacement. In the general case, the energy conservation equation considers both the conductive and convective contributions. Properly handling the convective term is one of the main challenges since it

yields an additional non-linear coupling between fluid flow and heat flux. In our analysis, this term is tackled by using a proper iterative linearization procedure. A novel formulation with the introduction of one additional variable (called pseudo-total pressure) is presented and studied. This additional variable ensures inf-sup stability and robustness with respect to locking phenomena in the quasi-incompressible limit. The robustness of the method with respect to the model's coefficients is one of the main targets of the first part of this work. To this aim, the proper handling of the non-linear term is carefully investigated. Indeed, it is possible to observe that for some choices of the model's coefficients the energy conservation equation becomes an advection-dominated equation. The question of developing discretization techniques that can handle these peculiar cases is addressed.

In the second part of the thesis, the problem of wave propagation phenomena in thermo-poroelastic media is studied. To this aim, the fully-dynamic THM problem is considered. In this case, the inertial terms

in all the three equations are included. The theory of wave propagation in porous media has been first presented by Biot and then developed by Carcione. Biot considered a fully-saturated porous media – as in our assumptions – and investigated the presence of three kinds of waves: two compressional (P) waves and a shear (S) wave (Fig. 1). The two P-waves propagate in different ways, the first one is a fast wave, while the second one is a slow wave, that is diffusive at low frequencies and is slower than the fast wave at high frequencies. In general, the introduction of the coupling with the temperature induces the appearance of an additional thermal wave (T), which is a slow diffusion wave.

In this work, *lymph* – a MATLAB library for the PolyDG approximation of multi-physics problems (Fig. 2) – is also presented. The numerical solution of coupled multi-physics problems, such as the THM coupling, is of crucial importance

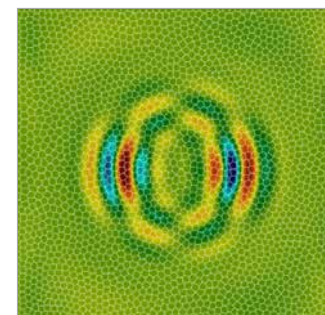


Fig. 1 - Wave propagation phenomena - vertical velocity field over the computational polygonal mesh



Fig. 2 - lymph logo.

nowadays and their numerical simulation is challenging due to their complex nature. Along with their intrinsic complexity, the accurate approximation of such problems through numerical methods often requires different constraints on geometric details, scale resolution, or local refinement of the computational mesh. The use of polytopal meshes has become increasingly popular as a solution for these numerical challenges, due to their flexibility in representing intricate geometries, interfaces, and heterogeneous media. Thus, a particular interest has been devoted to the development of numerical methods that can handle general grids. In this work, the discontinuous Galerkin finite element method on polytopal grids (PolyDG) is exploited. The use of this method offers numerous benefits when dealing with coupled problems: (i) accurate representation of complex geometries, (ii) flexibility in refinement and agglomeration strategies, (iii) ability to cope

with non-conforming interfaces, (iv) robustness with respect to heterogeneities of physical properties, (v) arbitrary-order accuracy.

In conclusion, in this thesis models and methods for effectively describing the wave propagation phenomena in thermo-poroelastic media are developed. Poly-DG formulations motivated by the fields of application are proposed and their semi-discrete analysis, showing hp-convergence results, is provided. Then, robust numerical schemes are developed, and their capabilities are assessed by addressing literature and physically-sound (heterogeneities, geothermal model problem, Fig. 3) test cases. Last, the important role of temperature in the behavior of shear waves is shown.

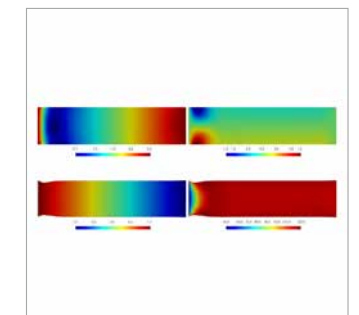


Fig. 3 - Geothermal model problem proof-of-concept - Numerical solution of the horizontal displacement (top-left), vertical displacement (top-right), pressure field (bottom-left), and temperature field (bottom-right).

## EMERGENCE OF MACROSCOPIC MODELS FROM THE BCS THEORY OF SUPERCONDUCTIVITY

**Andrea Calignano** - Supervisor: Michele Correggi

The most spectacular phenomenon observable for gas of particles at low temperatures is indeed quantum condensation. When this phenomenon occurs in a degenerate system of bosons (e.g., bosonic atoms), it is known as Bose–Einstein condensation; in a system of degenerate fermions (e.g., electrons), we refer to it as Cooper pairing. Depending on the bosonic/fermionic nature of the gas, a macroscopic fraction of particles or pairs of particles, which from now on we refer to as the condensate, behave in the same way. For example, in a gas of interacting bosons in a trap, one observes that the momentum distribution becomes very peaked around a point (typically zero momentum) at extremely low temperature as a consequence of the condensation of a large portion of the particles in the same (zero-momentum) one particle state. Fermions give another typical example in a metal: at room temperature, electrons move independently, while when the sample is cooled down below a specific critical temperature, the particles get bound in Cooper pairs by an attractive interaction due to the presence of the underlying ionic structure. A very striking consequence of quantum condensation is superconductivity (in a charged fermionic system, as

electrons in metals), a quantum mechanical phenomenon consisting of a sudden drop of electrical resistance and expulsion of magnetic fields occurring in certain materials when cooled below a critical temperature.

The BCS theory of superconductivity was formulated in 1957 by the American physicists Bardeen, Cooper, and Schrieffer. The BCS theory is already an effective description of the system, although still macroscopic, and can be derived at least heuristically from first principles description of a gas of electrons inside a metal under the assumption that the underlying ionic array mediates the quantum state in quasi-free and the interaction between the electrons. It was realized in the eighties that BCS theory applies to both BECs of tightly bound fermions and cases where the pairing mechanism is very weak. The only difference is in the nature of the pairing mechanism, but the consequent condensation phenomenon is the same.

A few years before the BCS description of superconductivity appeared, a phenomenological macroscopic explanation was provided by V.L. Ginzburg and L.D. Landau. In the Ginzburg-Landau

(GL) theory, the superconducting features of the sample are encoded in an order parameter, i.e., a complex wave function minimizing a suitable energy functional, which is supposed to approximate the free energy of the system. In presence of an applied magnetic field, the response of the system is encoded in another variable besides the order parameter which is the induced magnetic field: outside the sample and at its boundary, the magnetic field is given by the applied one, while inside the sample it is given by a self-generated field which minimizes the free energy functional.

For a dilute bosonic system the counterpart of the GL theory is the Gross-Pitaevskii (GP) one: when BEC takes place, all the particles of the gas behave in the same way and their one-particle wave function is shown to minimize a suitable nonlinear functional - the GP functional - which provides an effective macroscopic description of the phenomenon. For a fermionic gas, if the system favors the formation of stable bosonic molecules, the Gross-Pitaevskii theory description is obviously more appropriated. Indeed, if the physical regime under investigation is a BEC one, the mechanism behind the emergence

of collective behavior in the low-temperature Fermi gas is not the usual BCS pairing phenomenon but rather a condensation of fermionic pairs playing the role of bosonic molecules.

Since the derivation of the effective GP and GL theories has primarily addressed in translationally invariant systems or systems with weak external potentials that do not substantially influence the physics of the problem, a natural question arises: *is it possible to derive an effective theory in the case of physically relevant domains, with external potentials and/or in the presence of external magnetic fields that break translational invariance?*

Under the supervision of Michele Correggi, I worked on a non-translation-invariant systems, e.g., because of an external trap or the presence of a magnetic field. After presenting the physical models in which superconductivity can occur, we present a heuristic derivation of the BCS theory starting from the many-body quantum mechanics. To be more precise, we study a system of electrons trapped by an external potential  $\mathcal{W}$  with a polynomial asymptotic behavior. The two-body interaction  $V$  must be strong enough to bind two particles together. This is new since the primary references consider bounded domains or 3D space with bounded external potential. This implies more difficulties since one has to control the error generated by terms that do not decouple exactly (contrary to the explicit case of the harmonic oscillator).

Let us briefly present the setting: we set the length scale of the trap to be 1, while the microscopic interaction varies on a microscopic scale. The parameter  $h$  thus describes the ratio between the micro and macroscopic scales, and we study the limit for small enough  $h$  of the ground state energy of the BCS energy functional and any corresponding minimizer. We do not fix the number of particles a priori but study the grand canonical problem in the presence of a chemical potential. It is interesting to notice how, by fixing the average number of particles and exploiting the relation between the density of the microscopic problem and the macroscopic problem, one can realize how the choice of the chemical potential enforces the low-density regime.

In addition, we present different aspects of the semiclassical derivation of the GP theory from microscopic BCS. Also, the previous hypothesis on the bound state is left unchanged in this case. We noted that the perturbation of the Laplacian leads to some difficulties in controlling mixed error terms, but the derivation of the GP functional is mostly unaffected. This has been the first step towards studying the system in the presence of more physically relevant fields (i.e., anyonic particles carrying Aharonov-Bohm fluxes, strong magnetic fields, or self-consistent magnetic fields), along with physical boundary conditions. Thus, to get a more physically coherent understanding of the topic, we study a system

consisting of  $N$  fermions interacting pairwise in the presence of an external magnetic field. The added complexities arise from the fact that we now also consider an intense induced magnetic potential. This leads to studying the self-adjoint realization of a magnetic Schrödinger operator in a bounded domain with Neumann boundary conditions. In this setting, the behavior of a material sample can be read off from the properties of the minimizer of the GP functional, which now depends on two variables: the order parameter of the theory and an induced or self-generated magnetic potential. We consider a low-density fermionic system in a regime in which the parameter describing the intensity of the applied magnetic field is significantly large, and we fix the temperature at  $\beta T = 0$ . This time, to control some errors in minimizing the BCS functional, it is not sufficient to rely on a priori estimates based solely on the knowledge of the energy of the system. Instead, we require a more refined technique that considers a priori estimates one can derive from the Euler-Lagrange equations for the BCS functional. This approach allows us to conclude that the magnetic potential minimizing the BCS functional is close to the one describing a uniform magnetic field. Subsequently, we can obtain an expression for the BCS ground-state energy in terms of the ground-state energy associated with the GP functional up to small corrections.

# SDES ON HILBERT SPACES: SLOW-FAST SYSTEMS OF SPDES, STOCHASTIC OPTIMAL CONTROL WITH DELAYS AND APPLICATIONS TO ECONOMICS AND FINANCE

**Filippo De Feo** – Supervisor: Giuseppina Guatteri

Co-Supervisor: Salvatore Federico (Università di Bologna)

In this Doctoral Dissertation, we study two topics related to stochastic differential equations (SDEs) on Hilbert spaces. In the first part, after recalling classical results of the theory of SDEs on Hilbert spaces, we consider slow-fast systems of SDEs on Hilbert spaces, where the stochastic perturbations are given by Wiener processes with non-trace-class covariance operators, possibly. Such systems depend on a small parameter, representing the ratio of time scales between the two variables of the system, which are referred to as slow component and fast component, respectively. A crucial property of such systems is the so-called averaging principle, which states that, under appropriate assumptions, the slow component converges to the so-called averaged component, which follows the so-called averaged equation.

In this setting, we prove the  $1/2$ -strong order of convergence in the averaging principle, which is known to be optimal. We provide an application of such result to slow-fast stochastic reaction-diffusion systems perturbed by white noise.

In the second part of the Dissertation, after recalling classical results of the theory

of optimal control problems of SDEs on Hilbert spaces, we study optimal control problems of stochastic delay differential equations (SDDEs).

We consider two types of problems: one with delays in the state variable only and one with delays also in the control variable.

To regain Markovianity and use the dynamic programming approach, we rewrite the problems as optimal control problems of SDEs on a suitable Hilbert space. These infinite-dimensional formulations enable us to characterize the value function of both problems as the unique viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equations, which are fully non-linear second-order partial differential equations on a Hilbert space with an unbounded operator. For the problem with delays in the state variable only, we prove a partial regularity result of the value function with respect to a direction of particular interest. When the diffusion is independent of the control, such result allows us to define a candidate optimal feedback control. However, due to the lack of twice differentiability of the value function, we cannot prove a verification theorem using standard techniques based

on Ito's formula. Then, using a technical double approximation procedure, we construct regular approximants of the value function, which are viscosity supersolutions of perturbed HJB equations and are regular enough so that they satisfy Ito's formula. This procedure allows us to prove a verification theorem and construct optimal feedback controls. We provide applications of such results to Merton's type optimal portfolio problems, stochastic optimal advertising problems, and stochastic optimal investment problems with time-to-build.

## DETERMINISTIC AND STOCHASTIC ANALYSIS OF SOME DIFFUSE INTERFACE MODELS

**Andrea Di Primio** – Supervisor: Maurizio Grasselli

The doctoral dissertation focuses on the mathematical analysis of several deterministic and stochastic diffuse interface models. These systems provide a significant understanding of phase separation phenomena occurring in complex materials such as binary alloys, multicomponent fluid mixtures or polymer blends. The term “diffuse interface” refers to the fact that the interface between two different species is modeled as a region with positive thickness. After illustrating the preliminary material, i.e., the general motivation and the necessary mathematical tools, the main contributions contained in this dissertation are presented in two parts: the first deals with deterministic models, while the second is devoted to stochastic ones. However, in all the models discussed, we aim to work within the physical framework. Indeed, if the energy functional of the system is thermodynamically consistent, then it is possible to show that a solution carries physical meaning (i.e., that it takes values in a physically admissible range). The main method to achieve this goal involves modeling the potential energy of the system through a function whose derivatives establish

potential barriers: this approach ensures that any solution with finite energy must be confined within them. The first part of the dissertation contains the following contributions. First, we examine a system for electrically responsive polymer mixtures. In particular, we are interested in blends of a diblock copolymer and a homopolymer acting as solvent. The phase separation occurring between copolymer and homopolymer, as well as the one between the two blocks of the copolymer, allow the self-assembling of ordered structures at the mesoscopic level, that also turn out to be useful in the development of nanotechnologies. In this model, Maxwellian interactions play a significant role, as the application of external electric fields is one of the primary methods to induce pattern formation. The second model we analyze illustrates, in a slightly different setting, the other side of Maxwellian interactions. Therein, we consider the dynamics of magnetically responsive complex fluids, accounting for hydrodynamic effects and fluid–fluid interactions. Finally, the last deterministic system, models a ternary system where two fluids are blended

with a surfactant. These substances are able to reduce the interfacial tension between different fluids and allow the mixing of normally immiscible fluids (e.g., water and oil). In all the aforementioned models, the recurring mathematical challenges concern the existence of weak and strong solutions, their uniqueness and regularity properties, as well as their asymptotic behavior. The second part of the dissertation contains two last contributions on stochastic diffuse interface models. As the literature on the topic is still in its early stages of development, fundamental problems such as existence and uniqueness of solutions are open even for models whose deterministic version is widely understood. First, a coupled stochastic Navier-Stokes-Allen-Cahn system is analyzed. This can be regarded as a second-order relaxation of one of the prototypical diffuse interface models, i.e., the Navier-Stokes-Cahn-Hilliard system. The stochastic approach allows, contrary to the deterministic counterpart, to take into account unpredictable phenomena of thermal nature happening at the microscopic scale. However, stochastic models do not ensure a strong mass

conservation property, which is instead a key feature of certain physical systems. In order to recover it, it is possible to fine tune the choice of the random noise, although rendering the mathematical analysis more challenging. The effect of conservative noise on the Cahn-Hilliard and the Allen-Cahn systems is the subject of the last chapter of the dissertation.

# EFFECTIVE NUMERICAL MODELLING OF HILLSLOPE PROCESSES: SEDIMENT TRANSPORT AND LANDSLIDE RUNOUT

**Federico Gatti** - Supervisor: Luca Formaggia

Co-Supervisors: Carlo De Falco, Simona Perotto

Smart cities improve quality of life by implementing sustainable strategies through advanced technology and innovation. According to European Union and the Scientific Community, the formulation of effective methods to deal with climate change is one of the most important topics of research and technology. Methods means not only the identification of actions to face the causes of climate change, but mainly the development of technologies to adapt the society to climate changes. Hydrological instability, which results in floods and landslides, is certainly one of the effects of climate change with a major impact on people and the security of built environments. So, for cities in areas subject to hydrological instability, the development of tools capable of predicting extreme events is of great importance. Cities located at the downstream end of mountain basins are exposed to specific flood risks, in which sediment transport plays a significant role. Among hydrological instability mitigation strategies, we can consider the development of early-warning systems. They represent the only way to identify hazardous areas and to put in safe people and infrastructures in

case field stabilization strategies carried out by geologists become impracticable. In the present work, we deal with modeling hillslope processes. Hillslope processes refer to the various natural phenomena and mechanisms that shape the landscape and influence soil erosion, sediment transport, and landslides on sloped terrain. These processes are driven by factors such as rainfall, soil properties, vegetation cover, and topography. Here, we focus on two primary hillslope processes: i) **Runoff-Induced Erosion:** This process involves erosion of the ground surface due to the flow of water, often because of rainfall. When rain falls on a sloping surface, it can infiltrate the soil, increasing the water content at the surface. This excess water can lead to surface runoff, which, depending on the slope's steepness, can accelerate and cause erosion. Runoff-induced erosion is a critical factor in soil loss and sediment transport, especially during heavy rainfall events. ii) **Rainfall-Infiltration-Induced Landslides:** Landslides occur when a mass of soil or rock moves down a slope. Rainfall can play a crucial role in triggering landslides. When heavy rainfall

infiltrates the soil, it can increase the soil's weight and reduce its stability, making it more susceptible to sliding downhill. Landslides can vary in scale and behavior, from slow, gradual movements to rapid, catastrophic events. In this work, we focus on the dynamics of landslides, particularly the phase after initiation, which involves the movement and runout of landslide materials. These hillslope processes are complex and can have significant environmental and societal impacts, including soil erosion, sedimentation in rivers, flood risk, and damage to infrastructure. Developing accurate predictive tools for these processes is essential for managing these risks and making informed decisions in areas prone

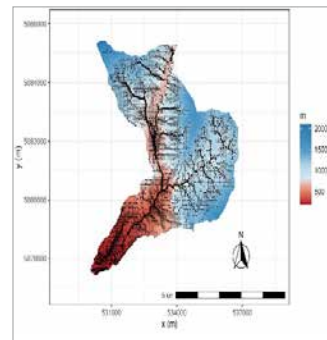


Fig. 1 - Example of river formation as output of the developed sediment transport model.

to hydrological instability. The first part of this work focuses on the hydrological risk of sediments and floods. We try to provide an efficient simulation tool capable of helping institutions in flood-prone areas during the decision-making process. A lot of modeling tools for this kind of process have already been developed, we can mention LISEM, WEPP, EUROSEM, SHESED, DHSVM, TOPKAPI, GEOTOP. Despite this, they suffer of several limitations. Indeed, they consist of a wide number of input parameters that need to be set somehow, e.g., with the help coming from field data provided by geologists, then they need the a priori subdivision of the basin region between river and slope region further without the possibility of simulating the potential formation of lakes proper of flood areas. Furthermore, they are mostly limited to the simulation of a single rainfall event either because of the limitations of the numerical scheme or because of the modeling of the infiltration rate law. Our model is designed to try to overcome these limitations.

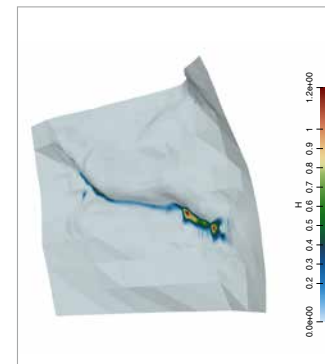


Fig. 2 - Isolines of the material height at the equilibrium state of a debris-flow landslide

The second part of the present work deals with the development of simulation tools for landslide prediction. The dynamics of a landslide is characterized by a broad range of velocity scales, each dominant in a particular phase of the event, from the steady creeping slip to a catastrophic avalanche, passing through the intermittent rapid slip. During these phases, the landslide displays different mechanical behaviors. Once the landslide is initiated, various behaviors take place, in particular a flow-like motion is typical of mud and debris flows, where the landslide either follows a viscoplastic behavior or the solid/liquid interaction is a key ingredient dominating the dynamic of the landslide material. In both cases, the overall process becomes advection dominated. This work focuses on the landslide runout dynamic and not on the landslide initiation phase. Due to the vast variety of landslides, we provide two numerical frameworks: a single-phase model for homogeneously moving slides, like mudflows, and a two-phase model for landslides characterized by non-negligible interactions between solid and liquid phases. In both cases, as happens for the proposed soil erosion model, we consider depth-integrated model equations considering that the horizontal length-scale propagation is considerably larger than the vertical one. Both schemes are a modification of the standard second-order two-step Taylor-Galerkin (TG2) scheme, by ensuring

well-balancing property and separation of scales between the advection and the source part. They maintain excellent parallel scaling performances (MPI implementation) proper of the TG2 method, while guaranteeing good accuracy when resolving internal stresses and bed friction terms, which represent potential sources of numerical stiffness. In particular, the presence of adaptive mesh refinement enables us to deal with large deformations that interest a landslide during its motion. All these properties make the proposed scheme particularly appealing and a valid alternative to numerical methods, such as Taylor-Galerkin and meshless schemes, already present in the literature to study such a kind of process.

# TOWARD A DYNAMIC APPROACH TO DECISION-MAKING IN CHRONIC ILLNESS: A STATISTICAL FRAMEWORK USING LONGITUDINAL REAL-WORLD DATA

Caterina Gregorio – Supervisor: Francesca Ieva

Co-Supervisor: Giulia Barbati

Non-communicable diseases remain the leading cause of mortality, morbidity, and healthcare cost burden worldwide. Decision-making in chronic diseases encompasses a wide range of prevention and treatment issues that health professionals at all levels face on an ongoing basis. Changes in individuals over time are of paramount importance in chronic diseases, as they affect their lives over a long period of time. During this time, some therapeutic decisions need to be reassessed as the individual's situation changes. An additional consideration that makes the management of chronic diseases such a challenging area today is the uncertainty associated with how the disease will evolve and how the medical decision will affect the individual, not only in the short term but also in the long term. The volume of healthcare data made available through digitization is now a reality. One of the most significant changes is the level of detail that can be provided about an individual patient. This is true both in terms of the heterogeneity of the information, as different data sources are integrated, and in terms of the temporal depth with which the data are

collected. However, this period is also one of great paradox: on one hand, the amount of data about each individual is increasing, but on the other, the use of this data for research purposes is still minimal. Longitudinal information in decision-making is often only taken into account from a qualitative point of view, with assessments sometimes being very subjective. From a methodological point of view, this is probably due to the various challenges in addressing real-world problems related to decision making employing a life-course perspective. The main focus of the Thesis is precisely addressing the difficulty of embedding observational data in an analytical framework to address specific questions related to the longitudinal

nature of decision-making in chronic illness. The recognition that personalized medicine in chronic diseases goes through the creation of new decision support systems that can help healthcare decision-makers synthesize and process the available information has been the driving concept behind all the research work presented. The Thesis contributes to extending previous research through a dynamic and lifetime decision-oriented approach and aiding the implementation of a novel evidence-based analytical paradigm for decision-making in chronic illness. To this end, we combine methods from the literature on causal inference, survival analysis, repeated measures, functional data analysis,

pharmacoepidemiology, and health economics to produce novel statistical approaches to address specific clinical problems and research questions. All contributions are motivated by research questions derived from real-world case studies. In addition, these problems are closely linked to various types of observational longitudinal data sources used, including disease registries, administrative health databases, electronic health records, and population-based studies. Importantly, the methods proposed tailored to the different medical problems related to decisions in chronic diseases, ranging from risk assessment to prevention and treatment management, lead to novel evidence in their respective field of research. The first part of the Thesis is dedicated to the development of dynamic risk-assessment tools that allow the exploitation of information contained in an individual's past data regarding biomarkers and treatment use while assessing their role in relation to adverse outcomes over time. The clinical application motivating this part of the Thesis concerns patients affected by heart failure and the monitoring biomarkers, with a particular focus on potassium and the use of Mineralokortikoid Receptor Antagonist. The data have been provided by Trieste Observatory of Cardiovascular Diseases, which links the Friuli Venezia Giulia Regional Administrative Health Database with the cardiological E-charts. The second part of the Thesis addresses the

effect of a treatment on time-to-event outcomes using observational data, taking into account the challenges posed by the time-varying dimension both in the covariates and in the outcomes. Specifically, a method to identify common treatment effect patterns over time to characterize the dynamic response of individuals to treatment, exploring survival neural networks and functional clustering. Is proposed The case study motivating this method is a continuation of the one considered in Part II. It concerns the problem of understanding in which patients with heart failure, stopping treatment with MRAs is harmful and when, on the contrary, it may be beneficial. The data source is again the Trieste Observatory of Cardiovascular Diseases. In addition in this part, a methodology to estimate the effect of a time-varying treatment on recurrent events and a terminal event based on marginal structural models is presented. The method is applied to a case study involving the estimation of the effect of beta-blockers on arrhythmia recurrence and survival in arrhythmogenic right ventricular cardiomyopathy. Data comes from the Trieste Heart Muscle Disease Registry. The third part of the Thesis addresses problems related to the evaluation of the long-term effect of an intervention using multi-state models and microsimulation. In particular, a framework to perform scenario analysis by estimating the conditional expected outcome after an intervention in a multi-state

setting to optimize its timing in a personalized manner is proposed. The real-world application is the identification of the optimal timing for haematopoietic stem cell transplantation in patients with myelodysplastic syndromes, starting from the clinical and genomic characteristics of the patients using data coming from an international disease registry. In addition, it is showed how microsimulation-based scenario analysis can be applied to build a framework able to generate reliable and timely evidence of the cost-effectiveness of drug treatments. The study concerns the cost-effectiveness of the use of PCSK9 inhibitor drugs on hyperlipidemia using the data provided by Trieste Observatory of Cardiovascular Diseases. Finally, the use of multi-state models and microsimulation in the context of quantifying the life-term impact of a disease is addressed. This chapter differs from the others as it moves the attention to population disease prevention. Specifically, we consider an application on cardiovascular risk. Data from two sites of the Swedish National Study on Age and Care have been used: Kungsholmen (SNAC-K) and Blekinge (SNAC-B).

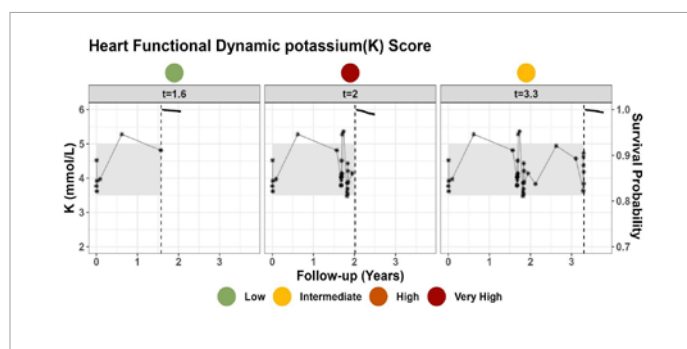


Fig. 1 - Example of dynamic prediction for one patient's HFDKS and corresponding 6-month survival probability

# A NEW COMPUTATIONAL APPROACH FOR FLUID-STRUCTURE INTERACTION OF SLENDER BODIES IMMERSED IN THREE-DIMENSIONAL FLOWS

Fabien Lespagnol - Supervisor: Paolo Zunino

The study of slender structures immersed in three-dimensional (3D) flows is of great importance in a spectrum of engineering disciplines, offering insights on phenomena such as vortex vibrations impacting submerged industrial structures or interaction of aquatic vegetation with the surrounding flow. Beyond engineering, its interest also extends to biomedical applications, such as the design of vascular stents or the study of micro-swimmers.

By exploiting the special geometric configuration of the slender structures, we modelled this problem using mixed-dimensional coupled equations in which the solid balance equations are formulated in a one-dimensional (1D) domain. Several challenges must be faced when dealing with these types of problems. From a mathematical standpoint, these include defining well-posed trace operators of co-dimension one and ensuring the accuracy of the solutions obtained with the mixed-dimensional formulation compared to a full three-dimensional formulation. From a computational perspective, the non-standard mathematical formulation makes it difficult to ensure the accuracy of the

discrete solutions with standard numerical approaches.

In the first part of the PhD, we established the continuous formulation of the fluid-structure interaction coupled problem using incompressible Navier-Stokes equations to describe the fluid dynamics and a linear Timoshenko beam model to model the dynamic response of the slender structure. These models were coupled with a mixed-dimensional version of fluid-structure interface conditions, combining the fictitious domain method with the projection of kinematic coupling conditions onto a finite-dimensional Fourier space via Lagrange multipliers. Furthermore, we developed a discrete formulation within the framework of the Finite Element Method and a semi-implicit treatment of Dirichlet-Neumann coupling conditions. We established the energy stability of the scheme and provided extensive numerical evidence of the accuracy and robustness of the discrete formulation, notably with respect to a fully resolved (ALE-based) model and a standard reduced modeling approach.

In the second part, we conducted a mathematical analysis of the approximation error of these methods, whether it was the modeling error arising from the mixed-dimensional formulation or the numerical approximation error resulting from the fictitious domain method with Finite Element approximation. We explored these aspects in two distinct simplified settings. We first considered an obstacle Poisson problem in a two-dimensional domain with non-homogeneous Dirichlet boundary conditions on small inclusions, obtaining a Poisson problem with defective interface condition. We then looked at a two-dimensional fluid-structure interaction problem where small particles were immersed in a Stokes flow, with reduced order interface coupling conditions. In both cases, after analyzing the existence of a solution of the reduced problem, we proved, for a fixed interface, its convergence towards the original full problem when the size of the obstacle tended towards zero, with a rate depending on the number of modes of the finite-dimensional space. In particular, our estimates highlighted the need to consider enough Fourier

modes to obtain convergence on the Lagrange multiplier, which was a key issue for the fluid-structure interaction coupling. Next, the numerical discretization of the reduced problem was analyzed. As is standard for these types of problems, the convergence of the solution is not optimal due to the lack of regularity of the solution, and the convergence exhibited a locking effect when the mesh size and the size of the inclusion were of the same order of magnitude. Thus, to resolve this issue, we proposed and analyzed a stabilized method and an enriched method.

One of the primary goals of mixed-dimensional formulations is to minimize the computational cost related to solving the solid problem while ensuring accurate computation of coupling conditions between the fluid and the structure. Pursuing the same objective, the most efficient approach for the time discretization of the fluid-structure interaction problem would be to adopt an explicit coupling scheme, that is, solving the fluid and structure only once per time step. However, for standard (Dirichlet-Neumann) explicit coupling schemes, a large fluid/solid density ratio

combined with a slender and lengthy geometry gives rise to unconditional numerical instability. Consequently, in the last part of the PhD, we introduced a Robin-Robin loosely coupled specifically designed for 3D mixed-dimensional formulation and proved its unconditional stability in a linear setting.

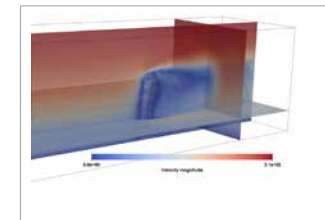


Fig. 1

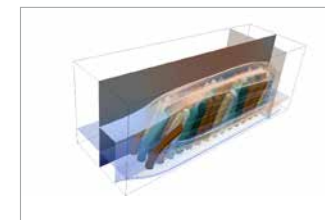


Fig. 2

# ON SOME NONLINEAR PARABOLIC AND ELLIPTIC EQUATIONS

**Troy Petitt** – Supervisor: Gabriele Grillo

Co-Supervisors: Matteo Muratori, Louis Dupaigne

The first part of my thesis concerns a weighted porous medium equation in the whole space. This equation models nonlinear degenerate diffusion processes where the inclusion of the weight represents an inhomogeneity in the mass density of the material in which the diffusion occurs. In particular, questions of optimal well-posedness are studied; that is, one aims to find the largest set of initial data for which the Cauchy problem is well-posed at least locally in time. From the mathematical perspective, the inclusion of a rough weight complicates the classical arguments and requires some new techniques and approaches. The first chapter is based on a published paper written with Matteo Muratori in which we study the problem of proving existence of solutions of the Cauchy problem given a locally integrable initial datum that may grow in an average sense at spatial infinity at a prescribed power rate. A recurring challenge of this equation is that the possibly rough weight breaks the usual translation invariance property of the problem. Such a property is used in the classical proof of existence in the particular step of proving

global quantitative smoothing effects of constructed solutions; qualitatively, this amounts to proving that the diffusion effects of the equation produce solutions which are immediately locally bounded out of initial data which may be only locally integrable. We are able to overcome this lack of translation invariance with significant effort by making use of a more general a priori estimate in the nonlinear Moser iteration method. The study of uniqueness, or proving that the constructed solutions are in fact uniquely determined by their initial data, is carried out by use of the so-called duality method, wherein a solution to a dual (backward-in-time) problem is used as a test function to the equation. The possible lack of regularity of the weight near the origin complicates this procedure, so we must remove a small ball and estimate normal derivatives on the boundary. In general, due to classical non-uniqueness results for the standard heat equation, it is unreasonable to expect uniqueness of general sign-changing solutions to hold without the requirement of an additional pointwise bound. The next chapter of the thesis

is based on a still-unfinished manuscript written with Matteo Muratori, Gabriele Grillo, and Nikita Simonov, concerning initial traces, local a priori estimates, and uniqueness of positive solutions to the same equation. Here we study general positive solutions without any further a priori information and we prove global integral estimates on the size of solutions; this information is then transferred to the existence, uniqueness, and estimates on the initial datum, which is in general only a Radon measure. As before, the classical proof of this result is based on translation invariance of the problem, which is no longer guaranteed, along with the delicate use of explicit special solutions. We propose a completely different method of proof, using the Green's function of the Laplacian as a test function and then using scaling properties of the equation, which, despite the rough weight, are still approximately valid for large spatial and time scales. With this estimate in hand, we can prove uniqueness for positive solutions even without the pointwise bound that is used in the first chapter in the context of uniqueness. The key

step to this aim is a uniform estimate on local solutions of the equation. By use of a delicate and highly nontrivial Moser iteration, we are able to prove such a local smoothing effect, which is optimal in space and even optimal in time in a certain range of parameters. Finally, the proof of uniqueness is proven by exploiting a Cauchy estimate in norms that take large data. The third main topic is the study of intermediate asymptotics for solutions, whose material comes from a submitted paper written with Matteo Muratori and Fernando Quirós. That is, we study how solutions evolve as time goes to infinity. In particular, we study the asymptotics of solutions whose initial data are not globally integrable, where much more is known. As is typical of such problems, the long-term diffusive effects of the equation pull general solutions towards simpler, self-similar special solutions, which act as attractors. In this case, we study initial data that decay at a prescribed power rate at spatial infinity, so the self-similar attractor is the solution whose initial datum is the corresponding pure power. Our first noteworthy result in this context is a new transition phenomenon for special solutions, in which the qualitative behavior of the time derivative changes depending on a certain parameter. This effect reveals a new regime where solutions are not integrable yet they seem to exhibit properties of the classical integrable

special solutions. Our main results however, are a pair of global convergence results of solutions to these special solutions, which take place in weighted Lebesgue spaces and even uniformly. Such results are new even for the classical unweighted porous medium equation, and requires radically changing the established approach of rescaling the equation down to its so-called blow-down limit in favor of an approach based on a priori smoothing and Cauchy estimates along with the study of convergence of approximate problems. It is also necessary to thoroughly study an ordinary differential equation. In the final chapter of the thesis, which is based on a published paper written with Louis Dupaigne and Alberto Farina, we study instead the classical Lane–Emden equation in the half-space and cones with homogeneous boundary conditions. Such an equation, which is a model for more general semilinear elliptic equations, sometimes exhibits the Liouville property – that is, the trivial solution is only solution of the equation in certain domains, ranges of parameters, and with certain boundary conditions. Indeed, we show that this is the case for classical and stable solutions posed in the half-space with homogeneous Dirichlet boundary conditions, regardless of the nonlinearity parameter on the right-hand side of the equation. By stable solutions we mean solutions that satisfy

a local energy condition which amounts to the positivity of the second derivative of the variational energy; in other words, the function is not just a critical point, but actually a local minimum. This interpretation gives heuristic support to the idea that stable solutions are the most natural and relevant of solutions. In fact, all of our results in this paper concern solutions that are merely stable outside of a compact set. Besides our main result on classical solutions in the half-space, we prove Liouville theorems on weak and non-negative solutions in the half-space, classical solutions in star-shaped cones contained in the half-space, and classical solutions in general cones in a certain range of parameters. We prove our results via a priori energy estimates, a monotonicity formula, and a Pohozaev inequality.

# MATHEMATICS OF PHASE SEPARATION: NONLOCAL INTERACTIONS, MULTI-COMPONENT SYSTEMS, EVOLVING SURFACES

Andrea Poiatti – Supervisor: Maurizio Grasselli

My doctoral thesis, carried out under the supervision of Professor Maurizio Grasselli, deals with the mathematical analysis of some phase-field systems modeling phase separation phenomena. Phase separation has been identified in many processes, from binary alloys for industrial purposes to Cell Biology. In particular, liquid-liquid phase separation has become a sort of paradigm to explain the formation of condensates in living cells, as well as to model the liquid behavior of the components of cell membranes, like phospholipids. In the phase-field models the interface separating two fluids is assumed to be a non-zero thickness region (diffuse interface). Therefore, the choice of the free energy, penalizing concentration variations mainly appearing at the diffuse interface, assumes a role of primary importance. Motivated by the classical literature, in this contribution we study both local models, generating from the Ginzburg-Landau free energy, and nonlocal models, in which the Helmholtz free energy additionally accounts for more generic long-range interactions. This dissertation is divided into three parts. In the first one we analyze some nonlocal

phase-field models for binary mixtures on fixed bounded domains, especially focusing on the analysis of a diffuse interface model for the motion of two globally immiscible, incompressible, viscous fluids with different densities and viscosities, i.e., the so-called nonlocal Abels-Garcke-Grün system. In the second part, motivated by the experimental evidence that the liquid-liquid phase separation in living cells involves mixtures composed by more than two fluid species, we concentrate on the mathematical analysis of multi-component local phase-field models on fixed bounded domains. In the third and last part of this dissertation, led by the growing interest in phase separation phenomena on elastic cell membranes, we study some local phase-field systems for binary mixtures on evolving closed surfaces. The evolution of the surface is a priori prescribed, and represents the evolution in time, for instance, of an elastic membrane. The common ingredient in our investigation is the adoption of the physically relevant logarithmic free energy density. This allows to show the existence of physical solutions, meaning that the order parameter, which in binary mixtures corresponds to

the difference of concentrations, gets only physically consistent values. This choice, however, makes the analysis more complicated, since handling the logarithmic potential is much harder compared to the choice of more regular functions. The main results presented here are mostly related to the existence of strong solutions, the regularization of weak solutions and the study of their longtime behavior. In particular, the leitmotif of most of the results is the validity of the so-called strict separation property. This means that the order parameter stays eventually (or even instantaneously) uniformly away from the pure phases, which is essential to achieve higher-order regularity for the order parameter itself, as well as for the study of its asymptotic behavior. Let us now present a brief summary of the main results in this dissertation. First, the nonlocal Cahn-Hilliard equation for binary mixtures on fixed three-dimensional domains is studied. By means of the De Giorgi iteration scheme we establish the validity of the instantaneous strict separation property, extending for the first time the known result in dimension two. This allows to achieve higher-order regularity

for the solutions and to prove that any weak solution converges to a single equilibrium. We then study the nonlocal Abels-Garcke-Grün system, by first proving the existence of global strong solutions in two-dimensional bounded domains and their uniqueness when the initial datum is strictly separated from the pure phases. Secondly, we show that any weak solution, whose existence was already known, instantaneously regularizes and converges towards a stationary solution as time tends to infinity. Furthermore, we demonstrate the continuous dependence of strong solutions with respect to general (not necessarily separated) initial data in the case of matched densities and unmatched viscosities. Concerning the multi-component phase-field models, we prove well-posedness and regularity results for the solutions to the multi-component local Cahn-Hilliard equation, generalizing the ones obtained by Elliott and Luckhaus in 1991. In particular, we establish the instantaneous uniform strict separation of solutions in two-dimensional fixed bounded domains. Then, we show the existence of a global (regular) attractor and we establish the convergence

of each trajectory to a single equilibrium also for three-dimensional bounded domains. Furthermore, we obtain analogous results for the multi-component conserved Allen-Cahn equation. In this case, by means of De Giorgi's iterations, we are able to demonstrate for the first time the validity of the instantaneous separation property also in dimension three, under some natural assumptions on the mobility matrix. Based upon this result, we additionally show the existence of an exponential attractor in both two and three dimensions. In conclusion, two-phase local Cahn-Hilliard type equations on evolving two-dimensional closed surfaces are considered. We prove some regularization properties of weak solutions in finite time, that allow us to show the validity of the uniform strict separation property from pure phases.